Multiple Constraint Choice Models with Corner and Interior Solutions

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Abstract

A choice model based on direct utility maximization subject to an arbitrary number of constraints is developed and applied to conjoint data. The model can accommodate both corner and interior solutions, and provides insights into the proportion of respondents bound by each constraint. Application to volumetric choice data reveals the majority of respondents make choices consistent with price and quantity restrictions. Estimates based on a single monetary-constraint choice model are shown to lead to biased estimates of the monetary value of attribute-levels.

keywords: multiple constraints, choice model, corner and interior solutions, quantity restriction

1 Introduction

Economic models of choice typically assume consumers maximize utility subject to a budget constraint. But, what if consumers are bound by multiple constraints? Examples included constraints on the maximum weight of packages, constraints on the size or volume of the product selected, and constraints on search time used for finding a good deal. If a consumer makes purchases subject to multiple constraints, models that only include a monetary budget constraint will misrepresent their behavior.

Consider, for example, consumers who are price and quantity constrained. These individuals respond to price changes for smaller packages sizes, but not for larger package sizes because of space constraints at their homes (see Bell and Hilber (2006) and Hendel and Nevo (2006)). The lack of response to deals for larger packaged offerings would translate into smaller estimated price coefficients, or larger error terms, in standard models of choice. When quantity or weight constraints are present, consumers may opt to purchase multiple units of smaller package sizes rather than larger packages, even when the per unit price of the larger package size is smaller. This behavior - selective response to variables such a prices - can be used to identify the presence of multiple binding constraints.

In this paper we develop economic choice models with multiple binding constraints, and contrast them to existing models. Thus, the nature of our intended contribution is methodological. We apply our models to volumetric conjoint data from a national study with the goal of determining packaging characteristics that maximize the quantity purchased. Our analysis indicates that the presence of multiple binding constraints biases estimates of the monetary value of attribute-levels if the model is estimated based on the assumption of single constraint only. We also find the multiple-constraint model leads to better in-sample and predictive fits to the data.

Models with multiple constraints have previously appeared in the literature, particularly in the context of travel demand and labor economics where consumers maximize utility subject to budget and time constraints. Two approaches have been proposed for analyzing these models. The first is based on the use of a "full price" variable obtained from substituting the time constraint into the budget constraint (with an associated value-of-time coefficient), and including the full price variable as a covariate in the specification. Becker (1965) used this approach to acknowledge the relationship between time and money in explaining how people allocate time, and Bockstael, Strand and Hanemann (1987) employed a similar substitution in their analysis of recreational demand. Kockelman and Krishnamurthy(2004) also applied this approach to the analysis of weekly travel demand. Hanemann(2006) offered a generalized version of this "collapsing" approach for an arbitrary number of constraints. The collapsing approach implicitly assumes a constant rate of trade-off among the constraints.

There are many situations where consumers face capacity constraints that are unlikely to change much and cannot be collapsed, either because the constraints are qualitatively distinct or because of fixed costs embedded in the conversion. For example, it is unlikely that consumers would bother to rent additional storage space to accommodate the purchase of large package sizes in many product categories. Moreover, we show below that the presence of multiple constraints can lead to interior solutions in models of near-perfect substitutes where indifference curves are linear (cf., Kim, Allenby and Rossi, 2002). Interior solutions cannot be explained by models with linear indifference curves with just one constraint. Thus, reducing the number of constraints and employing a “full price” variable is problematic when the constraints are not freely substitutable. Moreover, the use of latent consumptions occasions to explain multiple brand purchases, sometimes referred to as ”multiple discreteness” in the data (Dubè 2004), does not address the possibility of multiple constraints affecting consumer choices.

A second approach involves employing search procedures to identify the point of constrained utility maximization. The solution space associated with multiple constraints is partitioned into regions that condition on which of the constraints are binding, and one type of analysis proceeds by comparing solutions in each region to identify an overall maximum (see Hausman 1985, Moffitt 1986). More generally, Parizat and Shachar (2009) employ a simulated annealing algorithm over the entire solution space to identify a globally optimal point of utility maximization. The advantage of numerical solutions is that they do not require a continuous or differentiable solution space to identify the maximal point. The disadvantage is that statistical inference is hindered by relying on asymptotic approximations, and by not having access to parameters such as shadow prices (i.e., Lagrangian multipliers) that come from employing first-order conditions. We develop our model in a Bayesian framework that provides exact small-sample inference, and we demonstrate inferences within and among constraints that is not available when one only knows the point at which utility is maximized, and not properties derived from first-order conditions.

The remainder of this paper is organized as follows: in section 2 we develop choice models based on multiple constraints and compare it to existing models of discrete choice, and to models that allow for both interior and corner solutions. Statistical estimation of the model is discussed in section 3. The data are described in section 4, and section 5 presents the coefficient estimates from various models. Section 6 discusses implication of the model and examines predictions, and section 7 contains concluding remarks.

2 Model Development

We develop a multiple constraint choice model for an arbitrary utility function and two constraints. Extension to more than two constraints is straightforward. We begin with a utility function $U(\boldsymbol{x})$ that is maximized subject to budget and quantity constraints:

$$\begin{aligned} \max \quad & U(\boldsymbol{x}) \\ \text{s.t.} \quad & \sum_{n=1}^{N} p_n x_n + x_z = M, \qquad \sum_{n=1}^{N} q_n x_n + x_w = Q \end{aligned} \tag{1}$$

where $U(\boldsymbol{x})$ is assumed to be a quasi-concave, non-decreasing, one-time differentiable function of the vector of quantity demanded, $\boldsymbol{x}$. We assume that the utility function is weakly separable (see Deaton and Muellbauer, 1980) for the goods within the product category, and introduce outside goods x_z and x_w that proxy for other, non-category uses of money and quantity (i.e., space). This allows us to consider M as the maximum monetary allocation for the category, and Q as an upper limit for quantity. Here, p_n is the price of item n and q_n is the unit quantity of item n.

We introduced outside goods z and w with positive demand. That is, we assume that purchases made by consumers within a product category do not completely exhaust their budgetary allotment M nor their quantity allotment Q. There is always some residual budget and space available after a purchase, although the amounts may be small. The quantity demanded for z is x_z, with a price of 1 and a unit quantity of 0. So, the outside good z works for only budget constraint. Similarly, the quantity demanded for w is z_w with a price of 0 and a unit quantity is 1. So, the outside good w works for only quantity constraint.

We assume there are N items in a category, and R items are chosen. That is:

$$x_1, x_2, \ldots, x_{R-1}, x_R > 0, \quad x_{R+1}, x_{R+2}, \ldots, x_{N-1}, x_N = 0 \tag{2}$$

Finally, we define marginal utility as:

$$U_n(\boldsymbol{x}) = \frac{\partial U(\boldsymbol{x})}{\partial x_n} \geq 0 \tag{3}$$

Our approach to deriving the demand equations associated with multiple-constrained utility maximization is to derive the Kuhn-Tucker conditions by first forming the auxiliary equation that incorporates the constraints through the use of Lagrangian multipliers (λ, μ):

$$L = U(\boldsymbol{x}) + \lambda\{M - \sum_{n=1}^{N} p_n x_n - x_z\} + \mu\{Q - \sum_{n=1}^{N} q_n x_n - x_w\} \tag{4}$$

We assume the outside goods, z and w have positive demand. Thus, the two constraint are always binding, so λ and μ are positive.

Kuhn-Tucker conditions are derived that associate first-order conditions with observed demand:

$$\frac{\partial L}{\partial x_n} = U_n(\boldsymbol{x}) - \lambda p_n - \mu q_n \leq 0, \quad x_n \geq 0,$$
$$\text{and} \quad x_n \frac{\partial L}{\partial x_n} = 0 \qquad (n = 1, \ldots, N) \tag{5a}$$
$$\frac{\partial L}{\partial x_z} = U_z(\boldsymbol{x}) - \lambda = 0, \quad \frac{\partial L}{\partial x_w} = U_w(\boldsymbol{x}) - \mu = 0 \tag{5b}$$
$$\frac{\partial L}{\partial \lambda} = M - \sum_{n=1}^{N} p_n x_n - x_z = 0, \quad \frac{\partial L}{\partial \mu} = Q - \sum_{n=1}^{N} q_n x_n - x_w = 0 \tag{5c}$$

where, in equation (5a), $x_n(\partial L/\partial x_n) = 0$ is known as "complementary slackness", indicating that the constraint is binding whenever demand is non-zero. The expression for complementary slackness can be re-written as follows:

$$
\begin{aligned}
& x_n \frac{\partial L}{\partial x_n} = 0 \Rightarrow \\
& \quad \frac{\partial L}{\partial x_n} = U_n(\boldsymbol{x}) - \lambda p_n - \mu q_n = 0, \quad \text{and } x_n > 0 \qquad \text{(6a)} \\
& \quad \frac{\partial L}{\partial x_n} = U_n(\boldsymbol{x}) - \lambda p_n - \mu q_n < 0, \quad \text{and } x_n = 0 \qquad \text{(6b)}
\end{aligned}
$$

Thus, from (6a) and (6b) we see that positive demand is associated with an equality constraint from the Kuhn-Tucker first-order conditions, while zero demand results in an inequality condition. Similar to models with one constraint, the equality constraint leads to a density contribution in the model likelihood, and the inequality condition leads to a mass contribution.

3 Statistical Specification

A challenge in estimating models with multiple constraints is the presence of a heterogeneous consumer response to the constraints. Some respondents are price sensitive and others are not. Some are constrained by quantity, or time, or some other resource, and some respondents are bound by all the constraints under investigation. One aim of using models with multiple constraints is to make inference about the number, or proportion, of respondents bound by each. We employ a flexible statistical specification that can produce either corner or interior solutions, and allow for the possibility that consumers may not be bound by all of the constraints. Utility is expressed log-linearly:

$$U(\boldsymbol{x}_{kt}) = \sum_{n=1}^{N} \Psi_{knt} \log(x_{knt} + 1) + \Psi_{kz} \log(x_{kzt}) + \Psi_{kw} \log(x_{kwt}) \tag{7}$$

$$\Psi_{knt} > 0, \quad \Psi_{kz} > 0, \quad \Psi_{kw} > 0$$

where "k" denotes individual and "t" denotes the choice occasion. The "+ 1" in $(x_{knt}+1)$ allows for the possibility of corner solutions by translating the utility function so that indifference curves are not parallel to the axes. We assume that outside goods have positive demand and no corner solutions. The marginal utility associated with (7) is:

$$U_n(\boldsymbol{x}_{kt}) = \frac{\partial U(\boldsymbol{x}_{kt})}{\partial x_{knt}} = \frac{\Psi_{knt}}{x_{knt} + 1} \tag{8a}$$

$$U_z(\boldsymbol{x}_{kt}) = \frac{\partial U(\boldsymbol{x}_{kt})}{\partial x_{kzt}} = \frac{\Psi_{kz}}{x_{kzt}} \tag{8b}$$

$$U_w(\boldsymbol{x}_{kt}) = \frac{\partial U(\boldsymbol{x}_{kt})}{\partial x_{kwt}} = \frac{\Psi_{kw}}{x_{kwt}} \tag{8c}$$

Thus, the utility function is approximately linear for Ψ_{knt} large, and exhibits greater satiation for Ψ_{knt} small. We define Ψ as follows.

$$\log(\Psi_{knt}) = \theta_{kn} + \varepsilon_{knt}, \qquad \theta_{kn} = \sum_{d=1}^{D} \beta_{kd} a_{nd} \tag{9a}$$

$$\log(\Psi_{kz}) = \psi_{kz} \tag{9b}$$

$$\log(\Psi_{kw}) = \psi_{kw} \tag{9c}$$

where a_{nd} is the dth attribute of item n, D is the number of attributes, ε_{knt} is the type I extreme value distribution with scale parameter σ_k and β_{kd}, ψ_{kz},ψ_{kw} are parameters. Thus, our utility specification relates product attributes to the baseline marginal utility parameter θ, where more positive values of β results in greater marginal utility. We emphasize, however, that the choice of utility function is arbitrary - i.e., our method works for any valid utility function that is quasi-concave, non-decreasing, and one-time differentiable.

Substituting λ and μ from (5b) to (6a) and (6b), we obtain

$$\frac{\partial L}{\partial x_{knt}} = U_n(\boldsymbol{x_{kt}}) - U_z(\boldsymbol{x_{kt}})p_{knt} - U_w(\boldsymbol{x_{kt}})q_{kn} = 0, \quad \text{and } x_{knt} > 0 \tag{10a}$$

$$\frac{\partial L}{\partial x_{knt}} = U_n(\boldsymbol{x_{kt}}) - U_z(\boldsymbol{x_{kt}})p_{knt} - U_w(\boldsymbol{x_{kt}})q_{kn} < 0, \quad \text{and } x_{knt} = 0 \tag{10b}$$

where the time subscript is removed from the quantity variable to reflect a constant quantity associated with a specific choice offering. We comment on the identification of Q_k in this setting below. From (5c) we have,

$$x_{kzt} = M_k - \sum_{n=1}^{N} p_{knt}x_{knt} \tag{11a}$$

$$x_{kwt} = Q_k - \sum_{n=1}^{N} q_{kn}x_{knt}. \tag{11b}$$

Taking logarithms of (10a) and (10b), we have,

$$\varepsilon_{knt} = g_{knt}, \quad \text{and } x_{knt} > 0 \tag{12a}$$

$$\varepsilon_{knt} < g_{knt}, \quad \text{and } x_{knt} = 0 \tag{12b}$$

where

$$g_{knt} = -\sum_{d=1}^{D} \beta_{kd}a_{nd} + \log(x_{knt} + 1) + \log\left(\frac{\Psi_{kz}p_{knt}}{M_k - \sum_{n=1}^{N} p_{knt}x_{knt}} + \frac{\Psi_{kw}q_{kn}}{Q_k - \sum_{n=1}^{N} q_{kn}x_{knt}}\right) \tag{13}$$

Equation (13) comprises three terms. The first term is the deterministic portion of marginal utility that is also present in other models of demand, including the logit model. In the standard logit model with linear utility, this term is usually an intercept, or a linear function of product attributes and their levels. The second term is the logarithmic function induces diminishing marginal returns to the quantity demanded.

The last term in (13) is the contribution to utility from the outside goods that work uniquely for each of the constraints. As the budgetary allotment (M_k) or the quantity allotment (Q_k) increases, the quantity of the corresponding outside good increases by (11a) and (11b), and marginal utility decreases. This diminishes the effect of the variables in the numerator of the expressions that alter the quantity of the outside good purchased. Alternatively, as the allotments decrease the effects of the variables in the numerator increase. Thus, the allotments M_k and Q_k act in a manner that is similar to a price coefficient and quantity coefficient in a standard model.

For large values of M_k and Q_k, the budget and quantity constraints essentially become non-binding as respondents are unresponsive to variation in prices and quantities. Thus, our model nests more restrictive specifications with fewer constraints. Moreover, since M_k and Q_k are estimated parameters, it is often not necessary to estimate the coefficients Ψ_{kz} and Ψ_{kw} because they are nearly not identified, especially when M_k and Q_k are large. In the empirical analysis reported below, we set $\Psi_{kz} = \Psi_{kw} = 1.0$.

The assumption of extreme value errors results in a closed-form expression for the probability that R of N goods are chosen:

$$\begin{aligned}
\Pr(\boldsymbol{x}_{kt}) &= \Pr(x_{k,n1,t} > 0, \quad x_{k,n2,t} = 0, \quad n1 = 1, \ldots, R, \quad n2 = R+1, \ldots, N) \\
&= |J_{kRt}| \int_{-\infty}^{g_{kNt}} \cdots \int_{-\infty}^{g_{k,R+1,t}} f(g_{k1t}, \ldots, g_{kRt}, \varepsilon_{k,R+1,t}, \ldots, \varepsilon_{kNt}) d\varepsilon_{k,R+1,t}, \ldots, d\varepsilon_{kNt} \\
&= |J_{kRt}| \left\{ \prod_{i=1}^{R} \frac{\exp(-g_{kit}/\sigma_k)}{\sigma_k} \exp\left(-e^{-g_{kit}/\sigma_k}\right) \right\} \left\{ \prod_{j=R+1}^{N} \exp\left(-e^{-g_{kjt}/\sigma_k}\right) \right\} \\
&= |J_{kRt}| \left\{ \prod_{i=1}^{R} \frac{\exp(-g_{kit}/\sigma_k)}{\sigma_k} \right\} \exp\left\{ -\sum_{j=1}^{N} \exp(-g_{kjt}/\sigma_k) \right\} \qquad (14)
\end{aligned}$$

where $f(\cdot)$ is the joint density distribution for $\boldsymbol{\varepsilon}$ and $|J_{kRt}|$ is the Jacobian of the transformation from random-utility error (ε) to the likelihood of the observed data (x). For our model, the Jacobian is equal to:

$$J_{kijt} = \frac{\partial g_{kit}}{\partial x_{kjt}} = \frac{\delta_{ij}}{x_{kjt}+1} + \frac{\dfrac{p_{kit}p_{kjt}}{x_{kzt}^2} + \dfrac{q_{kit}q_{kjt}}{x_{kwt}^2}}{\dfrac{p_{kit}}{x_{kzt}} + \dfrac{q_{kit}}{x_{kwt}}} \tag{15}$$

where δ_{ij} takes 1 if $i = j$ and 0 otherwise.

Equations (13) - (15) are used to form the likelihood for the data. Across choice occasions for a respondent, the price of the choice alternatives change and M_k is statistically identified because its variation influences the impact of price changes on the choice probabilities. The quantity variable, q_{kn}, is not indexed by time and remains fixed across choice occasions when the quantity associated with a specific offering does not change over time. However, even in this case, there is variation in the outside good for quantity across the choices because of variation in x_{knt}, and non-linearities in the likelihood specification results in identified quantity constraints, Q_k. When only corner solutions are present in the respondent data, the likelihood is maximized when Q_k is set slightly greater than $\max_t \sum_{n=1}^{N} q_{kn}x_{knt}$. The likelihood profile for Q_k becomes more regular, with a well-defined mode, as the number of interior solutions increases.

In contrast to other models of multiple-discreteness, we do not associate the marginal utility of the outside good with its own error term (cf., Kim et. al. 2002, Kim et. al. 2007 and Bhat 2005). Instead, as shown below, we treat the unobserved constraints M_k and Q_k as parameters to be estimated. Our utility specification is similar to von Haefena and Phaneu (2003) and Bhat(2005, 2008). However, in these models, utility is assumed maximized subject to just one constraint, not multiple constraints.

More Than Two Constraints

Assuming there are S constraints. We introduce outside goods, $x_{c0}(c = 1, \cdots, S)$. The problem can be written as follows.

$$\begin{aligned} \max \quad & U(\boldsymbol{x}) \\ \text{s.t.} \quad & \sum_{n=1}^{N} p_{cn} x_n + x_{c0} = M_c \quad c = 1, \ldots, S \end{aligned} \tag{16}$$

Where M_c is the cth constraint and p_{cn} is the cth covariate(ex. price, quantity, ...) for good n. We assume that outside good c has coefficient 1 in cth constraint and 0 in other constraints. The remaining part of formulation is same as that of two constraints case. For example, the third expression in (13) would comprise S terms instead of two.

4 Empirical Application

Data are provided by a national beverage manufacturer wishing to understand the effects of packaging on demand using a conjoint study. Respondents were presented with choice between two product offerings of the same drink packaged in either bottles or cans, different container volumes (e.g., 1/2 liter bottles), different package sizes (e.g., 12 packs) that bundle the containers, and different prices. The goal of the study is to identify the effects of package attributes and prices on volume demanded. A list of product attributes is provided in Table 1. 3,478 observations from 282 respondents are available for analysis.

[Table 1]

Respondents were asked to choose from between two product offerings, and to indicate the expected monthly volume of the offering they expect to consume. Each respondent indicated their expected demand across 15 choice tasks involving combinations of eight different product configurations. Price was varied across all the choice scenarios so that none of the product descriptions appeared twice. In our analysis, information from the last choice task was reserved for predictive testing, and we expressed the expected monthly volume in terms of 100 fluid ounces. We comment on the use of this unit of measurement further below.

The data comprise both corner and interior solutions, where corner solutions indicate that just one of the two offerings are selected. The distribution of corner and interior solutions is presented in Table 2, indicating that about half the respondents purchase multiple packages of the beverage at least once in their choice history.

[Table 2]

A simple analysis of our data reveals that respondent demand is sensitive to price for smaller package sizes but not for large package sizes. A simple regression of demand on price (per ounce) reveals significantly negative coefficients for 4 pack and 6 pack offerings, but not for 12 pack offerings. The estimated regression lines, plotted in Figure 1, indicate support for the presence of a capacity constraint that reduces the attractiveness of purchasing large quantities of beverage for some respondents.

[Figure 1]

Our analysis involves the gains available from considering the presence of a quantity constraint in addition to a budget constraint. We constrained monetary (M) and quantity (Q) allocations as follows;

$$M_k > \max(\boldsymbol{p_{kt}}'\boldsymbol{x_{kt}}), \qquad Q_k > \max(\boldsymbol{q_k}'\boldsymbol{x_{kt}}) \tag{17}$$

We set $\Psi_{kz} = \Psi_{kw} = 1.0$ and $\sigma_k = 1$ for identification.

The likelihood for the data can then be expressed as:

$$L = \prod_k \prod_t \Pr(\boldsymbol{x}_{kt}) \tag{18}$$

Finally, all models are estimated with respondent heterogeneity specified by a multivariate normal distribution:

$$\boldsymbol{\Omega}_k = (\boldsymbol{\beta}_k', \log(M_k), \log(Q_k))' \sim \text{MVN}(\bar{\Omega}, V_\Omega). \tag{19}$$

Algorithms for model estimation are provided in the appendix A. 40,000 iterations of the Markov chain were used to generate parameter estimates, with the first 20,000 iterations discarded as burn-in.

5 Results

The range of the data is from a minimum of 64 to a maximum of $1,728$ ounces, with an average purchase volume of 208.4 ounces. Given the large numerical quantity of volume, we rescaled the demand data so that the offset value 1.0 produces indifference curves that are not nearly parallel to the axes at corner solution points. Thus the demand (x) is measured hundreds of ounces, with the demand for two 6-packs of 24 oz bottles is coded as $x = 2.88$ and $q = 1.0$. This rescaling was selected based on a grid search of alternative units of measurement – e.g., ounces, tens of ounces, hundreds of ounces, etc. – that maximized the model fit to the data.

Table 3 presents in-sample and predict model fits for different constraints - budget, quantity and both. The top portion of the table correspond to models involving one budget constraint, and the bottom portion of the table is model with multiple constraints. The results indicate that the multiple-constraint model fits the data best.

[Table 3]

Table 4 presents parameter estimates for the best-fitting model. The left side of the table reports the mean and standard deviation of the random-effect distribution for product attribute coefficients (Table 1) and the capacity. The right side of the table reports the covariance/correlation matrix.

[Table 4]

Table 5 presents parameter estimates for all models. The top portion of the table contains estimates for the budget constraint model, the middle portion is for the quantity constraint model, and the bottom portion for the multiple constraint model.

There is general agreement in estimates of type (β_1) and package size (β_4, β_5) for all models, but disagreement in estimates of container volume (β_2 and β_3). Thus, it appears that the absence of the capacity constraint biases estimates of attribute-level part-worths.

[Table 5]

6 Discussion

Incorporating multiple constraints into models of choice affords inferences about their relative importance in maximizing utility. We begin our discussion of the model results by classifying respondents into groups defined by posterior estimates of monetary and quantity budget, M and Q. We classify respondent into four groups, { high-M, high-Q },{ high-M, low-Q },{ low-M, high-Q },{ low-M, low-Q }. Medians of M and Q are used to define high-low groups. The median of M is 11.37(dollar) and Q is 6.21(100-ounces).

Since our model is based on constrained utility maximization, we can use the concept of indirect utility to investigate the relative importance of each constraint. Indirect utility, $V(p, q, M, Q) = U(x(p, q, M, Q))$, is defined as the maximum attainable utility given the binding constraints. The change in attainable utility as the constraints are relaxed is defined as the derivative of indirect utility with respect to M and Q:

$$\frac{\partial V}{\partial M} = \lambda \tag{20a}$$

$$\frac{\partial V}{\partial Q} = \mu \tag{20b}$$

That is, the Lagrangian multipliers λ and μ represent the change of maximum attainable utility due to the changes in the value of the budget allocation M and capacity Q constraints.

From (5b), we have

$$\lambda = U_z(\boldsymbol{x}) = \frac{1}{x_z}, \qquad \mu = U_w(\boldsymbol{x}) = \frac{1}{x_w} \tag{21}$$

Figure 2 displays average posterior estimates of λ and μ values for each respondent calculated according to the expression in (21). The average estimated value of λ is 0.20 and μ is 0.48, indicating that attainable utility will increase 2-fold more when capacity is increased by 100.0 ounces than when the budget is increased by \$1.00. This information is useful in deciding whether to pursue efforts in increasing household capacity through, for example, new packaging configurations as a means of increasing consumer welfare and increased sales. Table 6 shows segment-level shadow values. Low monetary budget segments have large λ and low quantity budget segments have large μ. This results indicate that small monetary budget people prefer price reductions and small capacity people prefer capacity improvements, as might be expected. The value of our model is that provides estimates of the value of relaxing each constraint.

[Figure 2]

[Table 6]

In addition to comparisons across the constraints, our model specification allows for the calculations of monetary equivalent (ME) values, or the willingness to pay, for the attribute levels (Ofek and Srinivasan 2002). The monetary equivalent of an attribute level is the dollar value that leaves the attractiveness of an offer unchanged. Since choices are related to the model parameters and data, including prices, through the Kuhn-Tucker conditions, we can use(10a) to obtain;

$$\frac{\Psi_n}{x_n + 1} = \lambda p_n + \mu q_n = \frac{p_n}{M - \sum_j p_j x_j} + \frac{q_n}{Q - \sum_j q_j x_j}. \tag{22}$$

Solving (22) for p_n, we obtain

$$p_n = \frac{A(M - \sum_{j \neq n} p_j x_j)}{1 + A x_n} \qquad A = \left(\frac{\Psi_n}{x_n + 1} - \frac{q_n}{Q - \sum_j q_j x_j} \right). \tag{23}$$

To obtain ME, first we define the null-level equivalent price p_n^{d0}, and the enhanced-level equivalent price p_n^{d1} for attribute-level d. Let p_n^* be a current price. We obtain ε_n in Ψ_n which satisfies (22). If the dth attribute-level is 0 - i.e., $a_{nd} = 0$ in Ψ_n, then let p_n^{d0} be p_n^* and p_n^{d1} be the equivalent price which is obtained from (23) with substituting $a_{nd} = 0$ for $a_{nd} = 1$. If the dth attribute-level is 1 - i.e., $a_{nd} = 1$, then let p_n^{d1} be p_n^* and p_n^{d0} be the equivalent price which is obtained from (23) with substituting $a_{nd} = 1$ for $a_{nd} = 0$.

Finally, we defined ME as follows:

$$ME = p_n^{d1} - p_n^{d0} \tag{24}$$

where positive values indicate the monetary value respondents are willing to pay for the enhancement provided by the attribute-level, and negative values indicate they would prefer the null-level of the attribute. A similar expression can be developed for quantity equivalents (QE).

Table 7 shows the ME of each attribute-level. The top row of Table 7 reports ME estimates for a budget-only choice model, and the bottom portion of the table reports estimates for the multiple constraint model. Since the quantity constraint model doesn't include price, we compare the budget constraint to the multiple constraint model. We find large differences in the monetary equivalents of the attribute-levels between the two models. The budget-only model indicates that respondents place less value on larger volumes and packages sizes, and actually prefer the 12 ounce container to the half-liter and 24 ounce size. The multiple-constraint model indicates the opposite - larger container and package sizes are preferred, but for the presence of binding constraints that make their purchase infeasible. Knowing that people prefer smaller containers because they have to, rather than because they want to, offers the chance to increase consumer utility through pursuits aimed at relaxing a person's capacity constraint. This insight is not available from the budget-only model, and the difference in the monetary equivalent estimates reported in table 7 are economically significant.

[Table 7]

Table 8 reports segment-level ME estimates for the multiple constraint model. Only the high monetary and quantity budget group has negative ME for half liter and 24-ounce volume. The two groups with low quantity budgets have larger MEs for larger container volume and package size. As we have shown in Table 6, respondents with smaller capacities have larger capacity shadow values. This results is consistent with the results of the ME analysis, pointing to large potential gains from relaxing the capacity constraint. Finally, we note that the groups with high monetary budgets express relatively strong preference for bottles over cans.

[Table 8]

A third insight afforded by our model is the relative rate of baseline marginal utility associated with the product offerings. Table 9 displays the expected value of θ in each of the eight product profiles examined in the study. Recall from (8a) and (9a) that marginal utility is related to product attributes through a log-linear specification where larger values of θ are associated with larger marginal utility and less satiation. We compute the posterior expectation of θ, by integrating over the respondent-level posterior distributions of part-worth estimates. Estimates for the profiles are found to vary from 1.16 to 2.10. Figure 3 displays indifference curves for profiles 1 and 8, the profiles associated with the extreme values of the θ. The figure indicates that good 8 is more likely to be consumed and is more likely to be associated with a corner solution.

[Table 9]

[Figure 3]

7 Concluding Remarks

In this paper we introduce a framework for developing choice models with multiple constraints. The framework is based on direct utility maximization, where choices are associated with model parameters through Kuhn-Tucker conditions. An outside good is introduced for each constraint in the model that is assumed to be independent of the other constraints. Thus, the outside good for a capacity constraint is assumed to cost nothing, and the outside good for the budget constraint is assumed to have no storage volume. The advantage of this formulation is that, so long as a constraint is binding, the quantity of the outside good can be determined by the observed demand for the inside goods and the constraint limits (e.g., M and Q) that are estimated as parameters.

The model is applied to conjoint data with the goal of understanding attribute-levels associated with volumetric demand. The model is shown to lead to improved in-sample and predictive fits. It also facilitates the evaluation of constraint importances through comparison of constraint shadow values through the Lagrangian multipliers (λ, μ). The Kuhn-Tucker conditions can be used to compute monetary and quantity equivalents (MEs and QEs) that can be used to translate the part-worth coefficients to a meaningful and common metric. These aspects of analysis are not available in direct-search models that do not rely on the Kuhn-Tucker conditions. They are also not available from models that move away from utility theory by, for example, introducing dummy variables and interaction terms to "pick up" aspects of the data, such as the price discounts of large quantity items being ignored because of quantity constraints. While more descriptive models are capable of fitting the data well, they do so at the expense of inference that employs shadow values and Kuhn-Tucker conditions. Finally, we show how the model can be used to understand the relationship between product attribute-levels and corner solution, which is important in multiple choice forecasting.

The presence of multi-constraints in consumer choice is, we believe, the rule rather than an exception. Consumers purchasing in nearly any product category must somehow deal with the wide array of choice offerings, including many brands and many sizes. One approach for modeling choices in this environment is to assume the presence of consideration sets (Bronnenberg and Vanhonacker 1996, Gilbride and Allenby 2004) where brand selection is made from a subset of available brands. A more structured approach is to investigate the constraints that are present that rule-out specific offerings as respondents maximize utility. For these constraints to be meaningful, it is necessary to allow for non-homothetic (i.e., non-linear) models of utility where constraint relaxation plays a non-trivial role.

All marketplace exchanges involve giving up resources and receiving benefits. Benefits are measured in terms of utility, with utility functions assumed to be quasi-concave and non-decreasing so that marginal utility is strictly positive. What is given up, however, diminishes the well-being of the consumer. Taken broadly, anything that leads to consumers being worse off should be measured through the constraints of an economic model, not through the utility function. Examples include the caloric content of food for people on a diet, the side-effects and potential interactions of drug prescriptions, and the down-side of drinking too much coffee (see also Hermalin and Isen, 2008). Since nearly all exchanges involve both approach and avoidance components, we believe this paper offers useful tools for developing and analyzing richer demand models in marketing.

Appendix A: Estimation Algorithm

Heterogeneity on individual respondents parameters is introduced hierarchically with the following distributions:

$$\Omega_k \sim \text{MVN}(\bar{\Omega}, V_\Omega)$$

We specify the prior distribution on hyperparameters as follows:

$$\bar{\Omega} \sim \text{Normal}(0, 100I)$$
$$V_\Omega \sim \text{IW}(\nu, \Delta)$$

where I is an identity matrix, IW is the inverted Wishart distribution, $\nu = (D+2)+4$ and $\Delta = \nu\mathbf{1}$.

A Gibbs-type sampler is used to draw from the conditional posteriors and distributions for heterogeneity.

1. Generate $\{\Omega_k, k = 1, \dots K\}$

A Metropolis Hastings algorithm with a random walk chain is used to generate draws of Ω_k. Let $\Omega_k^{(m)}$ be the mth draw for Ω_k and $s_k^{(m)}$ be the mth draw for s_k. The next draw $(m+1)$ for Ω_k is given by

$$\Omega_k^{(m+1)} = \Omega_k^{(m)} + \delta_\Omega$$

where δ_Ω is a draw from candidate generating density Normal$(0, 0.20^2)$. The probability of accepting the new draw, $\Omega_k^{(m+1)}$, is given by

$$\min\left[\frac{\exp[-1/2(\Omega_k^{(m+1)} - \bar{\Omega})' V_\Omega^{-1} (\Omega_k^{(m+1)} - \bar{\Omega})] \cdot \Pr(\boldsymbol{x}_k)^{(m+1)}}{\exp[-1/2(\Omega_k^{(m)} - \bar{\Omega})' V_\Omega^{-1} (\Omega_k^{(m)} - \bar{\Omega})] \cdot \Pr(\boldsymbol{x}_k)^{(m)}}, 1\right].$$

where $\Pr_h(\boldsymbol{x}_k)^{(m)}$ is the likelihood which is evaluated by using $\Omega_k^{(m)}$. If the new draw is rejected, let $\Omega_k^{(m+1)} = \Omega_k^{(m)}$.

2. Generate $\bar{\Omega}$

$$\begin{aligned} \bar{\Omega} &\sim \text{Normal}\left(B, \left(KV_\Omega^{-1} + 100I^{-1}\right)^{-1}\right) \\ B &= \left(KV_\Omega^{-1} + 100I^{-1}\right)^{-1}\left(KV_\Omega^{-1}\sum_k \Omega_k + 100I^{-1}(0)\right) \end{aligned}$$

3. Generate V_Ω

$$V_\Omega \sim IW\left(\nu + K, \Delta + \sum_k (\Omega_k - \bar{\Omega})(\Omega_k - \bar{\Omega})'\right).$$

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Table 1: Product Attributes

Attribute Name/Level	a_1	a_2	a_3	a_4	a_5
A. Type (a_1)					
A1. plastic bottle	0				
A2. can	1				
B. Container Volume (a_2, a_3)					
B1. 12 oz		0	0		
B2. 1/2 liter		1	0		
B3. 24 oz		0	1		
C. Package Size (a_4, a_5)					
C1. 4 pack				0	0
C2. 6 pack				1	0
C3. 12 pack				0	1

Table 2: Data Description

Number of Respondents	282
Number of Respondents with interior solutions	136(48.2%)
Number of one-alternative (corner) choices	2590(74.5%)
Number of two-alternative (interior) choices	888(25.5%)

Table 3: Model Comparison

Model	Log Marginal Density	
	In-sample	predictive
Budget Constraint	-7913.73	-707.10
Quantity Constraint	-8016.78	-733.93
Multiple Constraints	-7703.95	-706.24

Table 4: Parameter Estimates: Multiple Constraint Model

Attribute	Parameter	Mean	Covariance \ Correlation of Random Effects						
			β_1	β_2	β_3	β_4	β_5	$\log(M)$	$\log(Q)$
Can	β_1	-0.19(0.04)	0.22(0.03)	0.22	0.16	0.04	0.12	-0.25	-0.18
1/2liter	β_2	0.15(0.05)	0.06(0.03)	0.31(0.04)	0.64	0.51	0.59	-0.51	-0.55
24ounce	β_3	0.31(0.06)	0.05(0.03)	0.24(0.04)	0.46(0.06)	0.56	0.65	-0.43	-0.61
6pack	β_4	0.40(0.04)	0.01(0.02)	0.15(0.03)	0.20(0.03)	0.27(0.03)	0.67	-0.48	-0.54
12pack	β_5	0.74(0.07)	0.05(0.04)	0.31(0.05)	0.42(0.06)	0.33(0.05)	0.89(0.12)	-0.42	-0.67
Monetary budget	$\log(M)$	2.45(0.05)	-0.08(0.03)	-0.18(0.03)	-0.19(0.04)	-0.16(0.03)	-0.25(0.05)	0.41(0.04)	0.26
Quantity budget	$\log(Q)$	1.69(0.10)	-0.07(0.04)	-0.25(0.04)	-0.33(0.06)	-0.23(0.04)	-0.52(0.08)	0.14(0.05)	0.66(0.10)

Note: The numbers within the parentheses are the posterior standard deviations of random-effect parameters.

Table 5: Comparison of Estimated Parameters

Constraint	β_1	β_2	β_3	β_4	β_5	$\log(M)$	$\log(Q)$
Budget Constraint	-0.39	-0.17	-0.14	0.19	0.27	2.27	
	(0.04)	(0.04)	(0.05)	(0.04)	(0.06)	(0.04)	
Quantity Constraint	-0.23	-0.23	0.06	0.02	0.33		1.26
	(0.04)	(0.04)	(0.05)	(0.04)	(0.07)		(0.04)
Multiple Constraints	-0.19	0.15	0.31	0.40	0.74	2.45	1.70
	(0.04)	(0.05)	(0.06)	(0.04)	(0.07)	(0.05)	(0.10)

Note: The numbers within the parentheses are the posterior standard deviations of random-effect parameters.

Table 6: Segment level shadow values: Multiple Constraint Model

Monetary budget	Quantity budget	λ	μ
High	High	0.10	0.16
High	Low	0.12	0.64
Low	High	0.25	0.18
Low	Low	0.31	0.88

Table 7: Monetary equivalent of attribute-levels

Constraint Type	Type	Container Volume		Package Size	
	can	1/2 liter	24 oz	6 pack	12 pack
Budget Constraint	-0.54	-0.25	-0.13	0.25	0.36
Multiple Constraints	-0.46	0.36	0.68	0.79	1.08

Note:Figures are expressed as the median.

Table 8: Segment level monetary equivalent of attribute-levels
Multiple Constraint Model

Monetary budget	Quantity budget	Type	Container Volume		Package Size	
		can	1/2 liter	24 oz	6 pack	12 pack
High	High	-0.72	-0.54	-0.46	0.17	0.29
High	Low	-0.90	0.38	1.03	1.36	2.16
Low	High	-0.21	0.35	0.48	0.67	0.98
Low	Low	-0.13	1.23	1.85	1.79	2.00

Note:Figures are expressed as the median.

Table 9: Expected Baseline Marginal Utility E[θ] for Each Profile

Profile	Type	Container Volume	Package Size	Total Volume	θ
1	bottle	1/2liter	4pack	64	1.16
2	bottle	24oz	4pack	96	1.36
3	can	12oz	6pack	72	1.24
4	bottle	12oz	6pack	72	1.49
5	bottle	1/2liter	6pack	96	1.73
6	bottle	24oz	6pack	144	2.04
7	can	12oz	12pack	144	1.74
8	bottle	12oz	12pack	144	2.10

Figure 1: Price and Estimated Demand for Different Package Sizes

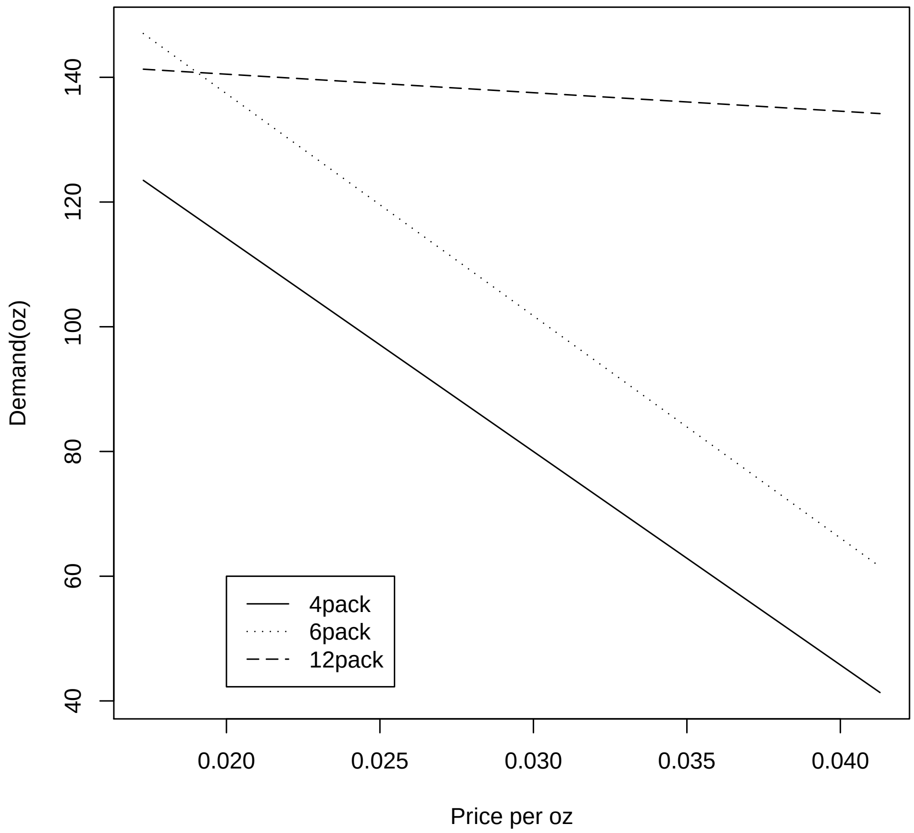

Figure 2: Individual-level λ and μ

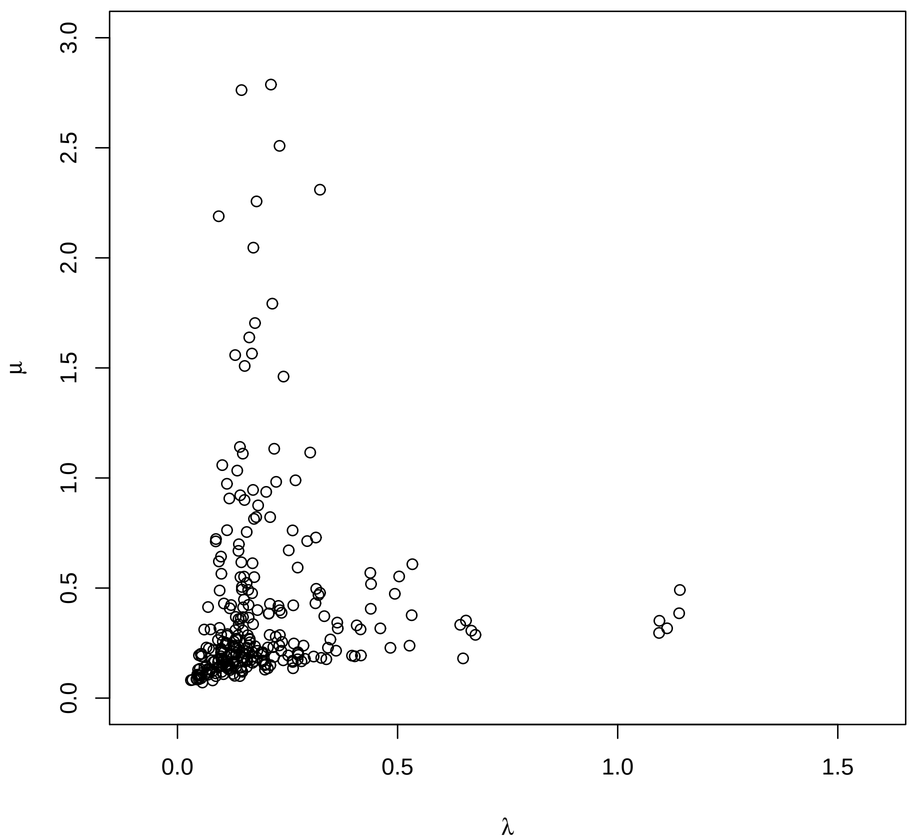

Figure 3: Indifference curves
Profile 1($\theta_1 = 1.17$) and Profile 8($\theta_8 = 2.10$)